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VAWT Linear BEM Theory

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Abstract: *Previous Vertical Axis Wind Turbine 'Vawt' models assumed the induced flow to be simply against the wind. Whereas in the better developed blade element momentum 'BEM' theory of Horizontal 'Hawt's', the induced flow is half the velocity change which is proportional to the lift vector normal to the apparent wind. High tip speed ratio brings the apparent wind to tangential, making the velocity change across the Vawt blades approach radial. Also not windwise are the lateral drift induced by the Vawt bound vorticity; and the reaction velocity to the torque in Hawt theory. That re-emerges as a productive second order crosswind in a new BEM Vawt model predicting the narrow operating peak of tangent and 'passive pitch' Vawts. This allows sound comparisons with the Hawt as explicitly solved, and the ideal variable pitch Vawt, as numerically optimised in a previous JPE paper. The new double pass solution incorporates the Prandtl tip correction and gives the cam shape for designing a benign and robust cyclic pitch Vawt.*

Keywords: Vertical Axis Wind Turbine, Blade Element Momentum, Darrieus, cyclic pitch, passive pitch

1. INTRODUCTION

The fixed tangent blade Vertical Axis wind Turbine (Vawt) was conceived in the 1920's by Darrieus whilst the theory of horizontal axis wind turbines (Hawts) was established by Glauert[1] in 1935. The Canadian government reinvented the 'Darrieus' or 'eggbeater' tangent blade Vawt in the 1960's and exclusively funded it, not any other Vawt let alone a Hawt, for the next 25 years. Sandia labs, but not the entire DOE, also specialised on the 'Darrieus' in the same period, leaving better documentation on the Internet. Since vertical blades have no gravity bending moment, Vawts are less limited in scale than Hawts are now becoming.

The analysis of the Darrieus by its proponents began with windwise induced flow of Froude's 1D actuator theory [1] which is also the high speed ratio limit [2] of Glauert's Hawt, now solved analytically. This new general solution[2] shows that the true limit in the Vawt case of blades moving obliquely to the wind is the induced flow being locally normal to the path and only windwise when averaged.

Wilson and Lissamen [3] averaged between the upstream and downstream Vawt cuts through each streamtube as if the lateral flows induced by the upwind and downwind cuts were equal and opposite and cancelled each other out at both cuts. This would be valid for a sufficiently elliptic blade path. Then the BEM finds that the common axial induced flow varies as the sine of the azimuthal angle. This optimistic assumption also ignored the vertical axial induced flow with the Darrieus troposkein shape as averaging to zero.

Others [4,5] 'refined' this model for the real circular path by allowing the sinusoidally varying windwise induced flow to be greater at the downwind cut, yet still neglected any crosswind or axial components. This “double (pass) multiple streamtube” Vawt was invariably solved numerically [4,5], which further obscures its soundness and its relation to Hawt theory. The aim of this paper is to formulate a Hawt-consistent two-pass Vawt model and solve it analytically......

2. FORMULATION

As in Fig 1 consider a section of a continuum of minute chord blades of solidity σ in a true wind $\underline{\mathbf{T}}$. For a Vawt or more properly a crosswind axis turbine, the axis of rotation is out of the page at a radius r in the unit direction $\underline{\mathbf{r}}$ shown. The blades move at velocity $XT\underline{\boldsymbol{\theta}}$ at speed ratio X times the windspeed T along a path at angle θ to the true wind $\mathbf{T}$, and ‘φ’ to the blade apparent wind ‘$\underline{\mathbf{W}}$‘. Call the velocity change they produce downstream $2\underline{\mathbf{I}}$, and their ¾ chord angle of attack to $\underline{\mathbf{W}}$, α.

Hawt BEM theory ignores any net pressure force on the stream tube walls and ends. (Detailed analysis [6] shows this is equivalent to assuming that all the crosswind velocity added at a Hawt rotor is ultimately dissipated.) It opposes the rate of change of momentum of the fluid density ρ passing through $\underline{\boldsymbol{\theta}}\, r\mathrm{d}\theta$, , to the Joukowski airfoil lift 'd$\underline{\mathbf{L}}$' with lift slope l of ideal from the net blade chord, $\sigma r\mathrm{d}\theta$

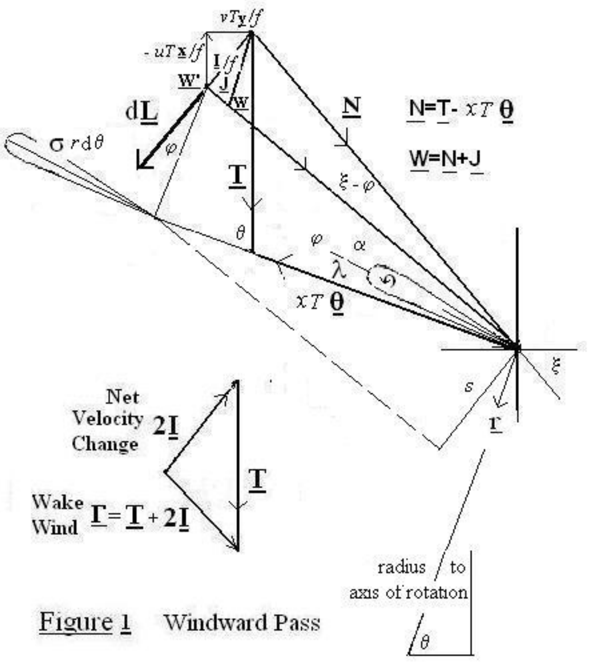

Figure 1 Windward Pass

$$\rho\ r\mathrm{d}\theta\ \ l\,(\sigma\pi W\sin\alpha\,\underline{\mathbf{z}})\ \mathbf{x}\ \underline{\mathbf{W}} \quad = -\mathrm{d}\underline{\mathbf{L}} = 2\underline{\mathbf{I}}\,\rho\, r\mathrm{d}\theta\ \underline{\mathbf{W}}\bullet\underline{\mathbf{r}} \quad \textbf{the linear lift BEM equation} \tag{1}$$

The bracket is the bound vorticity linear density along $\underline{\boldsymbol{\theta}}$, so the spanwise unit vector $\underline{\mathbf{z}}$ takes its sign by the right hand rule out of the page here on the windward pass. Now $\underline{\mathbf{W}}\bullet\underline{\mathbf{r}} = W\sin\varphi$ so

$$\underline{\mathbf{I}} = g\,(\underline{\mathbf{z}}\ \mathbf{x}\ \underline{\mathbf{W}}) \quad \text{so I} = g\text{W} \quad \textbf{the tangent Blade Element Momentum equation} \tag{2}$$

where most generally $g=\sigma C_L(\alpha)/\,4\sin\varphi$. For linear lift $g=\pi\sigma le/2=el\varpi$, $e=\sin\alpha/\sin\varphi$, and $\varpi=\pi\sigma/2=Bc/4r$ for B blades each of chord c. If the ¾ chords are tangential to the path as in the **fixed tangent blade** Vawt $\alpha=\varphi$ so $e=1$. To aid self-starting, “**passive pitch blades** “[4,7-9] use a balance between the centrifugal blade pitch moment varying as $(XT)^2\lambda$ due to a counterweight on an outboard arm to the blade, and the lift moment varying as $W^2\alpha$. Since $\lambda=\varphi-\alpha$ and $W\downarrow XT$, ignoring blade inertia and unsteady aerodynamics, their e is a constant fraction at all θ and large X . The Vawt analysis here will apply thirdly to the optimum design at X_d of the **robust cyclic pitch** Vawt[2] where e=½.

The first two invariant e’s with X signal narrow power bands, because the 1D momentum theory shows the power optimum I is $T/3$, but for tangent blades I changes almost linearly with its rotational speed Ω. That is when $W\rightarrow XT$, the interference $I/T\rightarrow$ $'a_0'=Xg=Xel\varpi$. In terms of the angular velocity Ω, $X=\Omega r/T$, so π and the radius ‘r’ of the rotor segment cancel, and $a_0=½BK\,l$ where $K=\Omega c/2T$. Forming the dot product of $\underline{\mathbf{I}}$ with the unit tangent $\underline{\boldsymbol{\theta}}$, the triple product identity gives

$$-\,\underline{\mathbf{I}}\bullet\underline{\boldsymbol{\theta}}\,/\,g = -\,\underline{\mathbf{W}}\bullet(\underline{\boldsymbol{\theta}}\ \mathbf{x}\ \mathbf{z}) = \underline{\mathbf{W}}\bullet\underline{\mathbf{r}} = W\sin\varphi = \underline{\mathbf{D}}\bullet\underline{\mathbf{r}} \quad \text{where } \underline{\mathbf{W}}=\underline{\mathbf{D}}-XT\,\underline{\boldsymbol{\theta}} \tag{3}$$

is the net flow relative to the ground. Thus the component of velocity change opposite to the motion is g times the normal velocity across the path. From the LHS of (1), the component of lift on the blade’s area $\sigma r\mathrm{d}\theta$ in the direction of travel is proportional to (3) and so multiplying by the blade speed gives the useful power contribution dP from $r\mathrm{d}\theta$, with the equivalences since $\underline{\mathbf{I}}\bullet\underline{\mathbf{W}}=0$

$$\mathrm{d}P/\,r\mathrm{d}\theta \;= -2\rho(\underline{\mathbf{W}}\bullet\underline{\mathbf{r}})\underline{\mathbf{I}}\bullet\underline{\mathbf{D}} = 2\rho T a_0\,(\underline{\mathbf{W}}\bullet\underline{\mathbf{r}})^2 \tag{4}$$

This squaring reflects the Vawt generating positive lift power by sin φ thrust resolution of its sin α lift. Equate the first term to the loss in flow kinetic energy as it crosses the rotor segment, again ignoring any pressure deficit in the wake. The change in velocity squared is $\underline{\mathbf{T}}\bullet\underline{\mathbf{T}}-(\underline{\mathbf{T}}+2\underline{\mathbf{I}})\bullet(\underline{\mathbf{T}}+2\underline{\mathbf{I}}) = -4\underline{\mathbf{I}}\bullet(\underline{\mathbf{T}}+\underline{\mathbf{I}})$, so this leads to $\underline{\mathbf{I}}\bullet(\underline{\mathbf{D}}-\underline{\mathbf{T}}-\underline{\mathbf{I}})=0$. Thus the component of the average induced flow $\underline{\mathbf{J}}=\underline{\mathbf{D}}-\underline{\mathbf{T}}$ in the direction of the velocity change $2\underline{\mathbf{I}}$ must be $\underline{\mathbf{I}}$, as in Fig 1. So the difference $\underline{\mathbf{J}}\text{-}\underline{\mathbf{I}}$ in the cross-sectional plane is parallel to $\underline{\mathbf{W}}$. Call the apparent wind based on $\underline{\mathbf{I}}$, , $\underline{\mathbf{W'}}$ which is thus in the same direction φ as $\underline{\mathbf{W}}$, but of different magnitude by $O(a_0T)$. Since the norm of $\underline{\mathbf{J}}$-$\underline{\mathbf{I}}$ should be less than ½ I or $T/6$ and $W \approx XT$ for $X>2$ it will be a good $O(ma_0)$ approximation to replace $\underline{\mathbf{W}}$ by $\underline{\mathbf{W}}'$ in (2).

Prandtl [1] considered the induced flow **field** around the vortex sheets trailing at $\mathbf{W}$ from the blades and thus spaced at $s= 2\pi\sin\varphi\, r/B$. They move downstream at self-induced 2**J** perpendicular to **W**. He found the induced flow **J** at the blades to be $\mathbf{I}/f$, f a function of the distance from the blade tip relative to s. This reduces the effective length of a straight Vawt blade by .442s.

3. SOLUTION FOR VAWT WINDWARD PASS AND HAWT

So in the high X limit, consider blade elements with $\underline{\mathbf{D}}$ just $\mathbf{J}= \underline{\mathbf{I}}/f$ plus the true wind $\underline{\mathbf{T}}$ in the $\underline{\mathbf{x}}$ direction at standard Vawt angle 'θ to the path. Then defining $m=1/X$, so $g=ma_0$; $H=1+g^2/f^2 \approx N^2/W'^2$; the tangent Vawt BEM equation (2) is linear in **I:**

$$I=f\underline{\mathbf{J}}=g\,\underline{\mathbf{k}}\ \mathrm{x}\ (\underline{\mathbf{T}}-\underline{\mathbf{S}}+\underline{\mathbf{J}})+O(ga_0T)\ \text{so}\ f\underline{\mathbf{J}}=g\,\underline{\mathbf{k}}\ \mathrm{x}\{\underline{\mathbf{T}}-\underline{\mathbf{S}}+g\,\underline{\mathbf{k}}\ \mathrm{x}\ (\underline{\mathbf{T}}-\underline{\mathbf{S}}+\underline{\mathbf{J}})/f\}=g\,\underline{\mathbf{k}}\mathrm{x}(\underline{\mathbf{T}}-\underline{\mathbf{S}})-g^2(\underline{\mathbf{T}}-\underline{\mathbf{S}}+\underline{\mathbf{J}})/f \qquad (5)$$

$$\therefore H\mathbf{I}=Hf\underline{\mathbf{J}}=(g\underline{\mathbf{k}}\,\mathrm{x}(\underline{\mathbf{T}}-\underline{\mathbf{S}})-g^2(\underline{\mathbf{T}}-\underline{\mathbf{S}})/f)=g\underline{T}\,\underline{\mathbf{y}}-a_0T\,\underline{\mathrm{r}}-g^2(\underline{\mathbf{T}}-\underline{\mathbf{S}})/f=-a_0T\,\underline{\mathbf{r}}+gT\underline{\mathbf{y}}+ga_0T\,\underline{\boldsymbol{\theta}}/f-g^2\underline{\mathbf{T}}/f+O(ga_0T) \qquad (6)$$

Dividing by T the non-dimensional Cartesian components u,v of $\underline{\mathbf{I}}/T$ are then

$$Hu= a_0\sin\theta+g^2/f+ga_0\cos\theta\ /f\ + O(ga) \qquad Hv= a_0\cos\theta+g-ga_0\sin\theta/f+O(ga) \qquad (7)$$

It is at first surprising that at fixed a_0, as speed ratio $X\to\infty$ and $g\to 0$, v does not vanish; but rather the net induced velocity becomes - $a_0T\underline{\mathbf{r}}$ directed outwards against $\underline{\mathbf{r}}$ (and so does not involve any power for lack of angle). This is a direct consequence of it and the lift having to be perpendicular to the blade apparent wind which high X brings towards tangential. Whilst the angle of attack and d$\underline{\mathbf{L}}$ decrease towards the sides, so does the normal flux, so from (1) the induced component stays constant.

The next and productive term $v=g=ma_0$ from $gT\underline{\mathbf{y}}$ is crosswind induction from the true wind component of $\underline{\mathbf{W'}}$. Then $ga_0T\underline{\boldsymbol{\theta}}$ reduces the contribution of the self wind to $\underline{\mathbf{W'}}$; and $-g^2\underline{\mathbf{T}}$ reduces that of the true wind. They will reduce the power, especially for the large a_0 of a single stage windmill (eg. a Hawt). Now the normal velocity at the blade is
$W'\sin\varphi(1+O(g))=\underline{\mathbf{W'}}\bullet\underline{r}=\underline{\mathbf{T}}\bullet\underline{r}\,(1-g^2/f^2H)-a_0T/fH+g\,T\,\underline{\mathbf{y}}\bullet\underline{r}/f\,H = T(\sin\theta-a_0/f-g\cos\theta/f)/H$ (8)
$a_0/f = q_0$ is just the total induced velocity at the blade. Then the reversal at small $\theta\approx q_0$, $\pi/2-\theta\approx q_0$ marks these as the points of the tangent edge streamlines dividing the Vawt passes. Alternatively the normal velocity can be formed from - $\underline{\mathbf{I}}\bullet\underline{\boldsymbol{\theta}}/g$ (3). W'^2 is $N^2/H=T^2(X^2+1+2X\cos\theta)/H$, producing a m second harmonic in φ adding to the first harmonics in (8). In the small m limit $W=XT$ and $\varphi\approx m(\sin\theta-q_0)$, a constant reduction from $m\sin\theta$ which can be recognised as the limit of ξ, the no-lift apparent wind angle Fig 1[2] . From equations (4) and

$$\mathrm{d}P/\,r\mathrm{d}\theta\ \ (1+O(g))=\ 2\rho fq_0T^3(\sin\theta-q_0-mq_0\cos\theta/)^2/H^2 \qquad (9)$$

Crude Hawt for small m ignores the H^2 and g factors to recover the 1D momentum theory $C_P=\mathrm{d}P/\mathrm{d}\,r\mathrm{d}\theta\ /\tfrac{1}{2}\rho T^3$ as $4fq_0(1-q_0)^2$ optimum at $16f/27$ for $a_0=⅓$. Exact Hawt $\underline{\mathbf{I}}=\underline{\mathbf{J}}$ leads to a cubic equation in q_0 for the optimum which has a triple angle solution which eventually reproduces the trisection theorems [2,10] that the best d$P/\,r$dθ comes from choosing $mq_0=\mathrm{I}/fW=\tan(\xi/3)$, so $\varphi=2\xi/3$, and further that $\lambda=\alpha=\xi/3$, $e=½\ \varpi=2\sin\lambda$ makes it robust against variations in m.

Whereas for constant e tangent or passive pitch Vawt blades, g must instead be constant whilst θ varies ξ, so one must first integrate over θ before optimising the coefficient of performance $C_P=P/\tfrac{1}{2}\rho T^3(2r)$, normalised on swept diameter. Note this will also average f over the variable s of the path.

$$C_P = \pi fq_0\,[1-8q_0/\pi+2q_0^2\,]/H^2 \quad (1+O(g)) \qquad (10)$$

with the approximate maximum at small m of C_P=.388f at q_0 =.265 averaged from 0°-180° vs .395f (2/3 of the Betz limit) for optimal pitch [2]. Taking $\sin\theta\approx\theta$ on the side zones where $q_0>\sin\theta$ gives a correction $-4q_0^4/3$ or C_P =.381f at q_0=.265. Strictly the highest allowable q_0 in the BEM is ½ at which the windward C_P is naively.356f, θ=0°-180° but actually .272f, θ=30°-150°.

Even with optimal variable pitch [2], the ideal two pass Vawt only approaches the Betz limit, and the tangent blade Vawt will not keep up.....

4. VAWT COMMON INDUCED FLOW MODEL

For the complete Vawt, Wilson & Lissamen [3] very simply ignored the separation of the windward and leeward passes so they had the same induced flows at each other, with cancelling signs of v, so the net $\underline{\mathbf{D}}$ is then just $(1-2u)T\underline{\mathbf{x}}$, changing solutions (7) (for $0<\theta<\pi$) to $u= a_0\sin\theta$ and $v=\pm a_0\cos\theta\pm(1-2u)g$

In this case the change in KE $2\rho\underline{\mathbf{I}}\bullet(\underline{\mathbf{T}}+\underline{\mathbf{I}})$ is $2\rho Ta_0(\underline{\mathbf{W}}\bullet\underline{\mathbf{r}})$, so (4) still holds. Fortuitously the small m average normal velocity component is unchanged, but the power average of the square of the normal velocity is changed, as well as doubled to account for the two passes. C_P is $4\pi fq_0$ times the average value of $(\sin\theta-2q_0\sin^2\theta)^2$. Defining the amplitude of the combined axial induced flow $A=2q_0$

$$C_P = \pi fA(\,1-16A/3\pi+3A^2/4\,) \qquad (11)$$

The maximum C_P=.554f at A=.401 puts the central induction perilously close to the value of ½ at which the centreline velocity in the wake is reversed and the model collapses. The above construction of a justification for ignoring the lateral induction contradicts considering the leeward pass to be in the wake of the windward, yet the "double multiple" streamtube method does both!

5. LEEWARD VAWT PASS

For Vawts the bound vortex cylinder, unlike the Hawt rotor disc, induces flow at the blades, producing a crosswind drift strongest on the rotor equator to the side in which the blades are advancing upwind. The Vawt bound vorticity from (1) is $gW\sin\varphi$ and its self-induced flow is the same $O(gT)$ or $O(m)$ less than the main radial induced flow a_0, and $O(mg \approx m^2 a_0)$ less than the self wind XT with which it is aligned at the peak power points.

As these $\theta = \pm\frac{1}{2}\pi$ points are a full diameter apart in the direction of the wind, it is more reasonable than common induced windwise flow to approximate the windward wake as fully developed at the leeward pass. Indeed this is the converse of the above treatment of the windward pass as being too upstream to be affected by the leeward wake. Let the average velocity drop at the leeward pass have components $2d$ and $-2n$ with induced flows at the blades d/f and n/f, respectively. Then the lee

$$\underline{\mathbf{D}}/T = (1-2u-d/f)\underline{\mathbf{x}} + (2v - n/f)\underline{\mathbf{y}} \qquad (12)$$

The algebra will be worked to $O(g^2)$ even though $O(g)$ is the magnitude of the uncertainty that the leeward wake's self-induced flow is exactly half its velocity change at finite speed ratio. As in Fig. 2 define a leeward $\phi = -\theta$, and note the reversed leeward circulation now gives $\underline{\mathbf{z}}$ into the page. Since some corrections and comparisons make the leeward e's different from the windward, consider a leeward 'h'

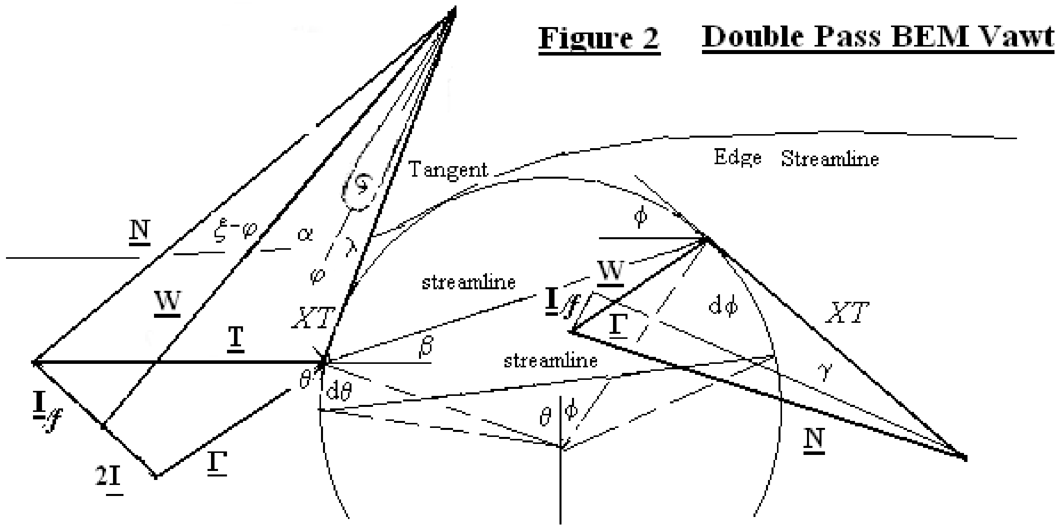

Figure 2 Double Pass BEM Vawt

in lieu of g and 'b_0'$= Xh$. As before, (2) $\underline{\mathbf{I}} = h(\underline{\mathbf{z}} \times \underline{\mathbf{W}})$ generates the simultaneous linear

$$d = b_0\sin\phi + hn/f - 2hv + O(gb_0) \qquad n = b_0\cos\phi - hd/f + h - 2hu + O(gb_0) \qquad (13)$$

Cross substituting these gives solutions analogous to the windward ones, but with the extra terms due to the $2u$ & $2v$ disturbances in the oncoming wind

$$(1+h^2/f^2)d = b_0\sin\phi + hb_0\cos\phi/f + h^2/f - 2h^2u/f - 2hv \quad (1+h^2/f^2)n = b_0\cos\phi - hb_0\sin\phi/f + h - 2hu + 2h^2v/f \quad (14)$$

Next forming the component $n\sin\phi - d\cos\phi$ of the leeward velocity change reacting to the leeward thrust, and then dividing out the h factor after leading term cancellation, gives by (3)

$$(1+h^2/f^2)\ \underline{\mathbf{W}}\bullet\underline{\mathrm{r}}/T = \sin\phi - b_0/f - h\cos\phi/f + 2\{u(h\cos\phi - \sin\phi) + v(h\sin\phi + \cos\phi)\} \qquad (15)$$

Substituting for the windward Vawt u and v from (7) gives H times the curly bracket as

$$H\{..\} = a_0(1+gh/f^2)\cos\theta{+}\phi + g(1+gh/f^2)\cos\phi + a_0(h-g)/f\sin\theta{+}\phi + g(h-g)/f\sin\phi \qquad (16)$$

At $a_0 = b_0$ and $\underline{\mathbf{I}} = \underline{\mathbf{J}}$ the ϕ average d is less than the θ average u by $2hg = 2m^2a_0^2$, and the mean n is less than mean v by about $4ha_0/\pi$. So the mean leeward I would be less than the windward and so the mean leeward $W' = I/g$ is less than the windward $W' = I/g$ all by a fraction $O(h)$.

The Hawt has a definite, if implicit, radial $\underline{\mathbf{J}}$-$\underline{\mathbf{I}}$ component expanding the streamtube as the wind is slowed axially. The corresponding vertical component will be assumed nil for a 2D Vawt, and mass conservation applied in the horizontal plane. Equating (8) x $d\theta$ to (15) x $d\phi$, allows in principle solving θ in terms of ϕ, for instance by expansion of the small difference in powers of a_0. The f intensification factors at the blades should not apply in this balance of average flows. Taking $1+h^2/f^2 \approx 1+gh/f^2 \approx 1+g^2/f^2 = H$, then no f factors appear in

$$(\sin\theta - a_0 - g\cos\theta)d\theta \approx d\phi\{\sin\phi - b_0 - h\cos\phi + 2a_0\cos\theta{+}\phi + 2g\cos\phi\} \qquad (17)$$

So to first order in a_0, expanding θ about ϕ gives

$$d[\sin\phi(\theta-\phi)] \approx d\phi(a_0 - b_0 + 2a_0\cos 2\phi + (3g-h)\cos\phi) = d\{\phi(a_0 - b_0) + a_0\sin 2\phi + (3g-h)\sin\phi\} \qquad (18)$$

$\beta = (\theta-\phi)/2$ is the mean deviation of a streamline inside the rotor and can be expected to vary from $v=g$ at $\theta=0$ windward to $2v-n=2g-h$ at $=0$ leeward so the average is indeed $(3g-h)/2$ so

$$(\theta-\phi)\sin\phi \approx (b_0 - a_0)(\pi/2 - \phi) + a_0\sin 2\phi + (3g-h)\sin\phi + O(a_0^2) \qquad (19)$$

confirming for $a_0 = b_0$, $m=0$ that the side point where $\sin\theta = a_0 = -\sin\phi$ is the tangent streamline of zero net flux. The approximation covers the optimal $a_0 \neq b_0$ to allow comparison of fixed and optimal[2] blades by exactly the same methodology. Expanding (15)

$$HW\sin\gamma = H\underline{\mathbf{W}}\bullet\underline{r}/T \approx \sin\phi - b_0/f + (2g-h/f)\cos\phi + 2a_0\cos2\phi - 4a_0\cos\phi\{a_0\sin2\phi + a_0(b_0-a_0)(\pi/2-\phi) + (3g-h)\sin\phi\} \quad (20)$$

Since the fractional difference in W's was found to be $O(h)$, this $W\sin\gamma$ smaller by only $O(a_0^2)$ than the windward means leeward γs become bigger as $X\downarrow$ (and $a_0\downarrow$ with it at fixed σ), as is clear must be so statically at X=0. For small m

$$\gamma \to m(\sin\phi - b_0 + 2\,a_0\cos2\phi - 4\cos\phi\{a_0^2\sin2\phi + a_0(b_0-a_0)(\pi/2-\phi)\} + O(m^2 a_0 \ldots a_0^3) \qquad (21)$$

So $\chi = ½\gamma$ and $\lambda = ½m(\sin\theta - a_0)$ gives the asymptotic robust fixed pitch cycle [2], suitable for a cam and different from the common first harmonic articulation from nose pushrods offcenter upwind. The $2a_0\cos2\phi$ terms makes this leeward γ less centrally than the windward φ if $b_0 = a_0$, but more on the sides for the extended angular range at the cost of range of φ. However the means of sin φ and sin γ are the same for small a_0, so their average f's will be approximated as the same. The final a_0^2 term has a negative mean from 0 to π making the mean γ a bit less than the windward φ, for a mean $\lambda-\chi$ robust ¾ chord noseout as indeed shown by Lazauskas's Fig. 7 to give a broad Vawt peak [4]

6. LEEWARD AND NET POWER

Ignoring terms above cubic in a_0, b_0; squaring and doubling (20) to form $H^2\mathrm{d}P/r\mathrm{d}\phi\ /2\rho T^3 b_0$ gives

$$(\sin\phi - b_0/f + (2g-h/f)\cos\phi + 2a_0\cos2\phi)^2 - 4a_0\sin2\phi\{a_0\sin2\phi + a_0(b_0-a_0)(\pi/2-\phi) + (3g-h)\sin\phi\} \qquad (22)$$

Integrating over ϕ from 0 to π by the average values of sinusoids with no average effect of the $(3g-h)/2$ skew in the center streamline, b_0-a_0 nor $3g-h$ and cancellation of the average a_0^2 terms gives

$$\text{lee } H^2 C_P = \pi f p_0\{1 - 8p_0/\pi + 2p_0^2 + (2g-h/f)^2 - 16a_0/3\pi\} + O(a_0^4) \qquad (23)$$

Where $p_0 = bo/f$ and again f becomes a mean. Note the (windward pass) endpoint correction was of the same $O(a_0^4)$. The final term is a negative effect of the leeward pass being in the wake of the leeward and will be conservatively approximated as $16q_0/3\pi$. Then the combined expression with windward C_P (10) is symmetric in $p_0 = q_0$ except the $(2g-h/f)^2$ term favouring q_0 at finite m and **I=J**, so otherwise expressing in terms of $A = p_0 + q_0$ and $2D = a_0 - q_0$ and f now an average over both passes

$$\text{Total } (1 + 1/4\, m^2A^2/f^2)^2\, C_P \approx \pi f A\{1 - 16A/3\pi + (½ + 1/4\, m^2)A^2\} + \pi D^2(6A - 32/3\pi) + O(A^4) \qquad (24)$$

A peak is at A=.348, D=0 giving C_P/f=.338 windward +.176 lee for a total of .514 vs. 16/25=.64 for a two stage Hawt and the Betz limit 16/27 for the optimal pitch or a Hawt[2]. This lower leeward power is inevitable because the power depends on the integral of the square of the normal velocity whereas the flux integrals of the first powers must be equal and are so by the extended leeward range making up for the lower peak leeward normal velocity. Numerically integrating (17) and (22) instead of the true expressions [2] lowered the C_P/f for optimal pitch by .013 from the Betz limit so fixed blades have a C_P/f=.514+.013=**.527** corrected for these approximations.

Manually changing the spreadsheet normal interference factors $a = \mathrm{I}/T\sin\theta$ [2] to give constant induction $A/2$=.174 windward and leeward concurs that C_P/f=.52 in the same approximations. The normal interferences are centrally .17 windward vs 0 optimal and .26 leeward vs always ⅓ optimal. The windward rises too slowly at first on the side but then too far instead of leveling at ½ whilst the leeward rises always about .1 more to sooner exceed its lower ideal ⅓. Or compare the constant 'induction' $a_0 = b_0$ here versus the ideal variable pitch induction μ_0 plotted in Fig 5 of [2].

C_P/f is .433 when the leeward theory becomes invalid with wake reverse flow at A=.5 or 1.44 the design X and the D^2 term still slightly negative, but C_P/f set to vanish at A=.76. Recall the windward pass- only optimum C_P/f = .38 at A is .53. Even with reversed flow the lift C_P cannot be negative so Clearly it must deflect upwards with A or with X for a given configured g. At A=1 when reversal has spread upwind to reach the center of the lee pass, the windward pass curtailed to 120° still has C_P/f .27. and say the total C_P/f may be .30 . Actually this is more extended than both the tangent and robust Hawt which both have C_P=0 at 3 times design X_d, though in-between Hawt pitches still have positive power with $\lambda > .145\varphi_\mathrm{d}$ avoiding reversed flow altogether [2]. Also the real Vawt has the higher drag penalty which grows remorsely as X^3.

{ }'s linear term in A which includes the interpass term is exactly the same as in the common interference model (11) and is generally indifferent as to how the induction is shared between the two passes. Yet the quadratic coefficient is ½ vs. Wilson's ¾ which explains the lowering of the optimum A from .401 to .348 with a narrower peak and more breathing room from reversed wake flow. Without the last streamtube expansion term in (23) there is an additional large $4a_0^2$ term in { } in (24) which doubles this coefficient and give excessive c_P's rising with A without peak!

Ignoring lateral induced flow and just solving the windwise component of (2) gives $u = a_0\sin\theta$ and $d = b_0\sin\phi$, which are not even the windwise components (7) and (14) of the real induced flow at finite m. Taking the lee pass net induced flow as $(2a_0 + b_0)\sin\phi$ [7] makes the above limiting quadratic coefficient come out to 15/16 for an extremely broad peak C_P/f of an unreal .605 at an improper A=.51; and is thus clearly unacceptable. Correcting this to downwind expanding $2a_0\sin\theta + b_0\sin\phi$ gives a coefficient of 9/16, still biased (blissfully ignoring a singularity in its θ-ϕ expansion even with $a_0 = b_0$ at sinϕ=0 which would likely cause numerical sensitivity.)

Thus none of these prior methods likely respect the Betz limit for optimal variable pitch, which would have been a basic test of their soundness. They all underestimate the reduction of angle of attack by a given a_0 by $\sin^2\theta$ or ϕ, so by ½ at π/4 . Thus a given α_{stall} is predicted at too high X on the sides. The neglected crosswind lateral reaction induced flow at finite X distorts the actual side angles of attack further. These basic inaccuracies completely undermine the attempts to treat dynamic stall hystersis effects in the typical VAWT code [5].

{ } in (24) has the value .47 at the peak, so the $\underline{\mathbf{I}}=\underline{\mathbf{J}}$ $O(m^2A^2)$ corrections on the two sides again virtually cancel. This can be attributed to no kinetic energy loss from no net wake crosswind flows as the windward and leeward pass crosswind flows cancel there. Likewise the finite m correction of the optimal pitch Vawt [2] should be weaker than for the optimal Hawt.

7. THE BREADTH OF THE POWER PEAK IN VAWTS AND HAWTS

Whilst the peak power deficit versus an ideal Hawt or optimally pitched Vawt of 10% is modest enough, the tangent & passively articulated blade Vawts have A varying inexorably with X and so a narrow operating power peak. The standard and simplest electric generator is synchrously tied to the grid which stiffly loads the windmill to run at virtually constant angular velocity Ω in fixed proportion to the grid frequency. Then it is vital that the wind turbine have a broad C_P, as X changes inversely with wind speed T [2]. Even with an ideal load cubic in Ω, the great gustiness in real winds means that the wind changes faster than inertia allows a wind turbine to change its Ω, so varying X momentarily.

The power drops off-optimum as $(\delta A)^2$ so as δX^2 for small variations δ away from the design values, whereas for a robust Hawt the drop is as δX^4 [2,10] At finite δX, the tangent Vawt angle of attack does not reduce quickly enough with X to avoid over-induction $A>$ ½ at X more than 44% design. In tangent Hawts[2] that leads to reversed wake flow and high, fluctuating aerodynamic loads that cause high noise, vibration and stress for low power, especially of arrays[2]. This zone that the Vawt frequents at high X's needs study, yet even tunnel flow visualisation seems lacking. Whereas with blade pitch λ only $.145\varphi_{\mathrm{d}}$ Hawts can **totally** avoid the turbulent windmill state and its high blade loadings and poor windfarm efficiency[2]

Vibration is a major Vawt problem. The change from uniform azimuthal flow and varying spanwise X that adds twist and root taper to the difficulty of making cantilevered Hawt blades, to spanwise constant chord, pitch, and induction for easily constructed and supported Vawt blades incurs azimuthal variation of blade lifts. Pairing blades has much more of a structural benefit for the fixed blade Hawt with less vibration penalty but has been risked more often in Vawt's! Avoiding the narrow power peak and high X reversed flow of fixed speed operation with more expensive variable speed generation makes the Vawt pass through a wide range of frequency Ω and its harmonics and risks resonances and fatiguing. An underestimated resonance forced the very large Cap Chat Darrieus to be run synchronously[5].

Robust Hawt blades fixed at halfway between the design apparent wind and the tangent keep their off-peak induction below optimal, for low blade loadings and good wind park efficiency [2]. At small X and induction, tangent blade Hawt's are the first to stall, whilst robust optimal fixed Hawt blades have higher inductions but smaller α, as numerically illustrated in [2]. The tangent blade Vawt can be converted to a broad robust peak and benign off-peak [2] by articulating the blade to follow the fixed $\lambda(\theta)=\varphi_d/2$ cycle of (21).

The ideal variable pitch [2] decreases e as $1/X$ to keep the induction A at the optimum and avoid the high X problems of high A, reverse flow, high blade and tower forces, and low C_P. Whirling a mass against a blade cam with quadratic centrifugal torque characteristic as $\Omega^2\lambda^2$would make α passively diminish asymptotically as $(\varphi-\alpha)^2$ and so as φ^2, and so e as $m^{\cdot}$. McDonnell Aircraft [11] calculated optimal α amplitudes that also gave e decreasing ultimately as $1/X$, but tragically did not test it or much else with their computer-articulated giromill.

A Hawt blade can be cambered for higher α_{stall} and maximum lift coefficient than a double-acting Vawt blade and fixed robustly for more induced flow to delay off-optimum stall . The centrifugal secondary flow [12] in a Hawt blade can increase the stall angle whereas the appendix describes measurements of net thrust in tangent Vawt-like pure pitching showing only dynamic mitigation of deep stall.

Also to be fair, Hawt blades can also pitch with very good cantilever bearings at their hub or effectively alter their outer span pitch with flaps, and need only do so on the time scale of T not $1/\Omega$. However they cannot alter their twist, so an ideal pitch-articulated Vawt [2]can be optimal over a wider speed ratio range than a variable pitch Hawt, slightly reversing the overwhelming advantage the Hawt has with fixed blades.

8.CONCLUSIONS

The common induced flow model has been rationalised and is proved to be optimistic versus separating the passes and including lateral induced flow. It is necessary to expand the streamtube to get sensible if still optimistic C_P with "double (pass) multiple" (streamtube) models. However neglect of lateral induced velocities completely undermines their inclusion of dynamic stall.

The Vawt operating power peak is inherently narrower than the Hawt's. Then at 1.44 the design X_{d} the fixed or passive pitch Vawt begins reversed wake flow which invalidates the leeward pass model. The actual flow needs to be observed and compared with the Hawt reversed flow which has been better observed even though Hawts can easily avoid this condition with minimal design pitch. So do quadratic cam centrifugal pitch articulation or the robust fixed pitch cycle, which also has half the intrinsic drag penalty. However the existing centrifugal pitch schemes do not avoid flow reversal. Like so much Vawt design they have suffered (fatally) from a general lack of impartial Hawt-correlated analytical insight, which if supplied at all by this paper will have fulfilled its purpose.

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APPENDIX 1 NOTATION

A the net interference factor p_0+q_0 at the blade for both passes
$a=I/T\sin\theta$ the local (radial) interference factor
$a_0 = Xg=XI/W$ representative of the non-dimensional actuator induction I/T at high X
B the number of blades
b blade semi-span
b_0 induction on the leeward pass for leeward e
c the local blade chord
C_D sectional drag coefficient
C_P - rotor coefficient of power / undisturbed kinetic energy flux through the swept area
d upwind component of leeward $\underline{\mathbf{I}}$ /T
f Prandtl tip correction factor
F real part of Theodorsen function
$e=\sin\alpha/\sin\varphi$ effective factor on blade chord
$H=1+g^2/F^2=N^2/W^2$
$g=e\varpi=I/W$ ratio of the half the velocity change to the apparent windspeed W windward
h leeward equivalent of g for leeward e and f
G complex part of Theodorsen function
$H=1+g^2/f^2$
$K=\Omega c/2T$ reduced frequency based on semichord & true wind
$k=\Omega c/2W$ reduced frequency based on semichord & apparent wind
l lift slope correction actor for thickness
m inverse speed ratio $T/\Omega r=1/X$
n crosswind (-$\underline{j}$ unit vector) component of $\underline{\mathbf{K}}$/T
P the dimensional power per unit length of span
$p_0=b_0/f$
$q_0=a_0/f$ the total induced flow at the blades
s spacing between wake vortex sheets
$t=2\pi r/B$ circumferential distance between blades center to center
T windspeed
u upwind component of $\underline{\mathbf{I}}$ /T
v crosswind ($\underline{y}$ unit vector) component of $\underline{\mathbf{I}}$/T
X local speed ratio $\Omega r/T$
W speed of the apparent wind
$\underline{r}$ unit radial vector <u>directed downwind</u>
$\underline{x}$ unit vector in the downwind direction
$\underline{y}$ crosswind unit vector towards side in which rotation is against the wind.
$\underline{z}$ unit vector along the blade span

I the induction, half the velocity change the blades produce downstream in their wake
J the actual induced flow velocity vector at the rotor
Γ the wake velocity **T**-2**I**
D=**T**+**J** the net airflow at the rotor relative to the ground
L airfoil lift vector
N the no-lift or nominal apparent wind **T**-**S**
T True undisturbed wind vector
W The net apparent wind vector **T**-**S** +**J**
α - ¾ chord angle of attack to **W**
α_{stall} the stall onset value of α
β mean deviation of streamline inside rotor
ϕ - Vawt downstream azimuthal angle (θ without streamtube expansion)
φ - The true or complete apparent wind **W** angle to the blade path (windward)
γ - leeward complete apparent wind angle to blade path
χ - leeward pass blade pitch
λ - windward blade pitch or angle of the 3/4 blade chord to the blade path
κ non-dimensional vertical gradient in the wind
ρ fluid density
σ - true local solidity = blade chords/circumference of blade travel
θ- windward azimuth angle
$\boldsymbol{\theta}$ - unit vector in azimuthal direction of increasing θ
$\boldsymbol{\phi}$- unit vector in azimuthal direction of increasing ϕ
ϕ - downwind azimuth angle
Φ - small inclination of blade span to vertical
ξ - The nominal or No-lift apparent wind **N** angle to the blade path
ϖ =πσ/2 =$Bc/4r$ half the net blade chord divided by the diameter
Ω- angular velocity Ω of rotation in radians per unit time
subscript $_d$ denotes design value
prefix δ denotes a small variation of

APPENDIX 2: CORRECTIONS

UNSTEADY AERODYNAMICS - In unsteady flow theory, oscillation of the ¾ chord angle of attack α about an apparent wind W, multiplies each harmonic by the linear operator F +iG, $F(k)$ and $G(k)$ are the real and complex parts of the Theodorsen function and the k are the multiples of $\Omega c/2W \lesssim c/2r$. At small k, 1-F≈k/2 and G=-4k. Neglecting the likely rotation of d**L** by (1-F)α only F in phase with α=φ generates mean forward thrust and power for the tangent e=1 Vawt. Such a primary F near 1 can be incorporated in l to mean less power for a given solidity, tipspeed, and drag. Hein's Fig 7.6 [13] verifies this reduction at k=.08 and .1 for sinusoidal pitch of a NACA 0012 at Re=3x10^5. At an optimum k=.055 Hien shows leading edge suction overcomes drag at 4.5° 0-peak amplitude, and the thrust mean coefficient peaks near the k=0 peak of .06 at 12° amplitude. Then it falls to .04 at 21° but dynamically avoids the k=0 zero at 24° and then rises above .06 above 28° amplitude . The persistence of a (drag) threshold at the (static) stall angle does not justify supposition of the thrust for the harmonics of the complex angle of attack variation found above .

DRAG: Generally one can correct the above ideal fluid power for real fluid profile drag $\frac{1}{2}\rho W^2BcC_D$. This rises at high speed ratios when the decrease $\delta C_P \approx 2\varpi\underline{(W/T)}^2XC_D \approx 2\varpi X^3C_D = -X^2C_DA/e$ Thus drag can be viewed as a X^2 increase in the negative linear coefficient in the A cubic (24), which favours lower A's, i.e. lower solidity at a given X. This term is about π less for a tangent Hawt segment where the swept width is 2πr vs. 2r, but the single pass requires twice the a_0 for the same A; whereas the Vawt always has the extra parasitic drag of any blade support arms and guys. In the Hawt the optimum X [10] is a tradeoff between the drag penalty increasing as X^3 and the ideal lift power (at optimal 1/3 angular interference little shifted by drag [10]) rising with X, as the swirl energy lost in the wake decreases. Here the two opposite passes leave little crossflow energy in the wake, so the rise with X is much sooner due to unstalling. Designing the peak angle of attack as the static stall value of say 12° gives.X=3.9e . There δC_P is .11e for an average C_D of .02 up to stall; so the robust cyclic peak is not only broader than the tangent, it is higher by about 10% due to half the drag penalty. Then the design ϖ =$Bc/4r$ is .045/e^2 which gives Hien's optimal k=$c/2r$ =.055 for B=2.5/e^2

BLADE ANGLING & ARCING- If there is no wind the flow is purely the curved self wind. Just as for a cambered blade the zero lift setting is when the blade midline at the ¾ chord point is aligned with the flow. Then the common blade setting of midchord tangent can be considered a built-in nose-in angle of attack of $c/4r$ windward for e>1 and -$c/4r$ leeward for e<1..

The latter's dominance of tangential self-wind at high X has suggested to many curving tangential finite chord Vawt blades to match the curvature of their path to reduce their ultimate form drag, though still air quantification is lacking. Then the only normal velocity component towards the blade surface is from the real wind. As above the zero lift reference for arced blades is the tangent at the ¾ point so that is again the evaluation point for φ and the reference point defining the blade θ. The camber would extend the angle of attack range on the lee pass vs the windward which is the opposite of that needed as stall first appears in this high X theory, and extra lift due to camber does not have leading edge suction for a thrust benefit. But it might help powering up through low X.

Experiments (5,14) have shown that ¾ chord tangency is in fact close to optimal for a symmetrical blade Vawt and definitely superior to the common naive tangency at midchord.

3D: The neglected vertical axial expansion flow reduces the downwind pass induced flow, to improve its performance a bit. At the tip keeping A constantly optimum implies the chord must vary as $1/f$ which resembles a elliptical rounding of the tip. The effective loss of span is $.221\ 2\pi r \sin\varphi/B$ at each Vawt tip for a net correction of $1.4\ r\sin\varphi/Bb$.

Vawt proponents generally used the aspect ratio induced drag correction of single wings. The minimum and uniform self-induced velocity of an elliptic blade semispan b due to a plane sheet of vortices trailing to infinity is $L/\ 2\pi\rho Wb^2$. For B blades circumferential distance 't' $\approx 2\pi r/B$ apart the LHS of (1) gives $L=2\rho\, 2bt\ W \sin\varphi\, \mathrm{I}$, also perpendicular to <u>**W**</u>. So a minimum for the ratio of the blade induced velocity to the continuous actuator's is $2r\sin\varphi\ /Bb$, so again keeping the total at the blade optimal vs. T, the power is reduced by this fraction, at least 40% greater than the Prandtl loss.

Certainly to avoid the low lift and high drag of stall for starting one wants low aspect ratio $2b/c$ and an elliptical planform. Crudely the Prandtl tip half-"ellipse" is about 5 times the loss of span long so its ideal planform is elliptical when the average $1.4rm(2/\pi - a_0)/Bb \approx .2$. Since the product of this and $2b/c$ is $.7e(2/\pi - a_0)/a_0 \approx 1.86e$ at the design optimum a_0 of .172. , the aspect ratio $2b/c$ should be about $9.3e$, promising low for self-starting with cyclic pitch with its smaller angles of attack in the first place

WIND GRADIENT - Robust Hawt's [2] only lose optimality to $O(\kappa^4)$ due to the gradient $\kappa T/r$ in the true wind T, and with three blades their rotor moments are smooth to $O(\kappa^2)$. Considering constant chord Vawt spanwise effects, whilst a_0 is independent of r, it does decrease vertically with the wind gradient losing optimality as $O(\kappa^2)$. So ideally the Vawt chord c should vary as T vertically.

More important are the gradients in wind cubed $3\kappa T^{\,3}/r$ and the speed ratio $-\kappa X/r$. Upright Vawt blades have lowest X and highest $\mathrm{d}P/\mathrm{d}s$ at their tops in the strongest wind. Therefore stall first appears dramatically by dropping the heavy blade loadings at the top of the Vawt, promoting its super vibration. Inclining straight Vawt blades outwards by $\Phi = \kappa$ for uniform X seems a minimum precaution.

And inclining matches the Vawt capture area better to the windpower shear. In the common induced flow model [3], inclination does not change the <u>average</u> induced flow in the wind direction nor the maximum power per unit vertical distance *ie Cp***.** (However again the drag correction is as the inclined length.) For a straight blade of a set length, the inclination Φ (top out) for the most swept power at a given mean radius r is $\Phi = \kappa b^2/r^2$ and inclining by $\Phi = \kappa$ for constant X gains C_P over upright by the fraction $\kappa^2(b^2/r^2 - \frac{1}{2})$.

Of course Vawt's capture twisted winds better than big Hawt's. Those even lose power as the cube of the cos of the wind yaw, as their power-consuming yaw motors act slowly in response to wind shifts.